DYNAMICAL BOREL–CANTELLI LEMMA FOR HYPERBOLIC SPACES

BY

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ABSTRACT

We prove that almost every (resp. almost no) geodesic rays in a finite volume hyperbolic manifold of real dimension n intersects for arbitrary large times t a decreasing family of balls of radius r_t, provided the integral $\int_0^\infty r_t^{n-1}dt$ diverges (resp. converges).

1. Introduction

In this paper, we study the shrinking target problem for the geodesic flow on hyperbolic manifolds of finite volume. Let V be a finite volume hyperbolic manifold (possibly complex, quaternionic or Cayley hyperbolic) of real dimension n. Write T^1V for the unitary tangent bundle over V, π: $T^1V \to V$ the canonical projection, $(\phi^t)_{t\in\mathbb{R}}$ the geodesic flow on T^1V, μ the Liouville measure on T^1V, and d the Riemannian distance on V. We are interested in the Liouville measure of the unit vectors that generate a geodesic ray that keeps intersecting a shrinking family of balls for arbitrary large times. We show

THEOREM 1: *Let $(B_t)_{t\geq 0}$ be a decreasing family of closed balls in V (with respect to the metric d), of radius $(r_t)_{t\geq 0}$. Then for μ-almost every (resp. μ-almost no) v in T^1V, the set*

$$\{t \geq 0 : \pi(\phi^t v) \in B_t\}$$

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is unbounded (resp. bounded) provided $\int_0^\infty r_t^{n-1}dt$ *diverges (resp. converges).*

General criterions implying the dynamical Borel–Cantelli alternative have been proved in various settings; see Chernov and Kleinbock [CK01], Conze and Raugi [CR98] and [CR03], and Dolgopyat [Dol04].

As a consequence of Theorem 1, a generic geodesic ray approaches the point p at a speed which is of the order of $t^{-1/(n-1)}$. More precisely,

COROLLARY 1: *For all* p *in* V *and* μ*-almost every* v *in* T^1V,

$$\limsup_{t\to+\infty}\frac{-\log d(p,\pi(\phi^t v))}{\log t}=\frac{1}{n-1}.$$

This is the analog of Sullivan's *logarithm law for geodesics* (see [Su82], and also [KM99] in the context of symmetric spaces), when one considers balls in the manifold rather that horospherical neighborhoods of one cusp. When V is compact and of negative curvature, Hersonsky and Paulin [HP01] provided sharp estimate of the Hausdorff dimension of geodesic rays that accumulate on a point p in V exponentially fast, depending on the exponent.

Theorem 1 is deduced from the following general proposition. Let $(\phi^t)_{t\in\mathbb{R}}$ be a flow acting on a probability space (X,μ), preserving μ, and such that the flow is ergodic with respect to μ. Let $F=(f_t)_{t\geq 0}$ be a family of functions $f_t\colon X\to\mathbb{R}$, such that $F\colon [0,+\infty[\times X\to\mathbb{R}_+$ is measurable. Such a family will be said to be **decreasing** if for any positive real numbers $s\geq t$ then $f_s\leq f_t$, and **positive** if $f_t\geq 0$ for any $t\geq 0$. Denote by $L^p(\mu)$ the space of measurable functions of integrable p-power, endowed with the classical L^p norm. We will write

$$S_T[F](x)=\int_0^T f_t(\phi^t x)dt,\quad I_T[F]=\int_0^T\left(\int_X f_t d\mu\right)dt.$$

PROPOSITION 1: *Let* p *be in* $]1,+\infty[$, $F=(f_t)_{t\geq 0}$ *be a measurable, positive and decreasing family of functions such that* f_t *is in* $L^p(\mu)$ *for all* $t\geq 0$. *Assume that* $\lim_{T\to+\infty} I_T[F]=+\infty$, *and that* $S_T[F]/I_T[F]$ *remains bounded in* L^p *norm as* T *goes to infinity. Then* $S_T[F]/I_T[F]$ *converges weakly in* $L^p(\mu)$ *to the constant function* 1, *and there exists* $c\in[1,+\infty]$, *such that for* μ*-almost every* x *in* X, *we have*

$$\limsup_{T\to+\infty}\frac{S_T[F](x)}{I_T[F]}=c.$$

Proposition 1 fails for $p=1$; see [CK01], Proposition 1.6 and remark after. Note that it is possible to deduce from Fatou's Lemma (see [KM99], 2.3, and

Lemma 3) that if F is a measurable, positive and decreasing family of $L^1(\mu)$-functions such that $\lim_{T\to+\infty} I_T[F] = +\infty$, then we have

$$\liminf_{T\to+\infty} \frac{S_T[F](x)}{I_T[F]} \leq 1.$$

As in the Borel–Cantelli Lemma, the divergence case is the difficult one. We show that for some functions f_t on T^1V related to the balls B_t, $S_T[F]/I_T[F]$ remains bounded in L^2 norm; then apply the preceding proposition to conclude. The arguments to bound the L^2-norm are purely geometrical, and do not involve the rate of mixing of the geodesic flow, unlike [KM99], but however make use of the very particular symmetries of rank 1 globally symmetric spaces. In the case when one replaces f_t in the proof by the characteristic function of the unitary tangent space of the ball B_t, one obtains the following

PROPOSITION 2: *Let $(B_t)_{t\geq 0}$ be a decreasing sequence of closed balls in V, of radius $(r_t)_{t\geq 0}$. Then for μ-almost every (resp. μ-almost no) v in T^1V, the set*

$$\{t \geq 0 : \pi(\phi^t v) \in B_t\}$$

is of infinite Lebesgue measure (resp. finite Lebesgue measure) provided $\int_0^\infty r_t^n dt$ diverges (resp. converges).

As a final remark, we wish to underline that the exponent $n-1$ in Theorem 1 is related to the dimension, and not to the critical exponent of the fundamental group, as one can see by considering the non-real hyperbolic spaces.

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2. Proof of Proposition 1

We will need the following sequence of Lemmas. We refer to [Ru75] for facts about measure theory.

LEMMA 1: *Let $F = (f_t)_{t\geq 0}$ be a measurable and positive family of functions. Then if*

$$I_\infty[F] = \int_0^{+\infty} \left(\int_X f_t d\mu \right) dt < +\infty,$$

we have that $S_T[F]$ tends everywhere and in L^1-norm toward a positive and L^1 function.

Proof: Put $S_\infty[F](x) = \lim_{T\to+\infty} S_T[F](x)$, which exists because $S_T[F]$ is increasing in T. From Lebesgue's monotone convergence Theorem, $S_\infty[F]$ is in $L^1(\mu)$ and

$$\int_X S_\infty[F]d\mu = I_\infty[F].$$

Lebesgue's dominated convergence Theorem allows us to conclude that the convergence occurs in L^1-norm. ∎

We recall that, if two reals $p > 1$ and $q > 1$ are such that $1/p + 1/q = 1$, the two Banach spaces $L^p(\mu)$ and $L^q(\mu)$ are dual to each other, and that a sequence $(g_n)_{n\geq 0}$ in $L^p(\mu)$ converges weakly to g means that for every h in $L^q(\mu)$, $\int_X g_n h d\mu$ tends to $\int_X gh d\mu$ as n tends to infinity.

LEMMA 2: *Let $F = (f_t)_{t\geq 0}$ be a measurable, positive and decreasing family of functions, and assume that f_0 is in $L^p(\mu)$. If $I_\infty[F] = +\infty$, then every weak limit in $L^p(\mu)$ of $S_T[F]/I_T[F]$ as T tends to infinity is the constant function equal to* 1.

Proof: Let h be a weak limit in $L^p(\mu)$ of a subsequence $S_{T_n}[F]/I_{T_n}[F]$. Let $t > 0$ be fixed. We wish to prove first that $h \leq h \circ \phi^t$. We have

$$S_{T_n+t}[F](x) - S_{T_n}[F](x) = \int_{T_n}^{T_n+t} f_s(\phi^s x)ds \leq \int_0^t f_0(\phi^s(\phi^{T_n}(x)))ds.$$

So this difference is positive and bounded in L^p-norm. Dividing it by $I_{T_n}[F]$, we obtain that $S_{T_n+t}[F](x)/I_{T_n}[F]$ converges weakly to h. On the other hand, we have

$$\begin{aligned} S_{T+t}[F](x) - S_T[F](\phi^t x) &= \int_0^{T+t} f_s(\phi^s x)ds - \int_0^T f_s(\phi^{t+s}x)ds \\ &= \int_0^t f_s(\phi^s x)ds + \int_0^T (f_{s+t} - f_s)(\phi^{t+s}x)ds. \end{aligned}$$

Since f_t is decreasing in t, we have

$$S_{T+t}[F](x) - S_T[F](\phi^t x) \leq \int_0^t f_s(\phi^s x)ds,$$

and when dividing by $I_{T_n}[F]$ and replacing T by T_n, we obtain

$$\frac{S_{T_n+t}[F](x)}{I_{T_n}[F]} - \frac{S_{T_n}[F](\phi^t x)}{I_{T_n}[F]} \leq \frac{S_t[F](x)}{I_{T_n}[F]}.$$

Let E be the set of x such that $h(x) \geq h(\phi^t x)$. We can write

$$\int_E \frac{S_{T_n+t}[F](x)}{I_{T_n}[F]} - \frac{S_{T_n}[F](\phi^t x)}{I_{T_n}[F]} d\mu(x) \leq \frac{\int_E S_t[F](x)dx}{I_{T_n}[F]},$$

and since the characteristic function of the set E is in $L^q(\mu)$, we obtain as $n \to +\infty$ the inequality

$$\int_E h(x) - h(\phi^t x) d\mu(x) \leq 0.$$

From the definition of E, this means that $h \leq h \circ \phi^t$ almost everywhere. Because of the invariance of μ and that h is in $L^p(\mu)$, this implies that $h = h \circ \phi^t$ almost everywhere. Since the flow is ergodic with respect to μ, h is constant (almost everywhere). On the other hand, $\int_X S_T[F]/I_T[F] d\mu = 1$, so $h = 1$ almost everywhere. ∎

Lemma 3: *Let $F = (f_t)_{t\geq 0}$ be a measurable, positive and decreasing family of functions, and suppose that f_0 is in $L^1(\mu)$. Assuming $I_\infty[F] = +\infty$, then the function from X to $[0, +\infty]$*

$$L(x) = \limsup_{T\to+\infty} \frac{S_T[F](x)}{I_T[F]}$$

is constant almost everywhere. Moreover, the function from X to $[0, +\infty]$

$$l(x) = \liminf_{T\to+\infty} \frac{S_T[F](x)}{I_T[F]}$$

is constant almost everywhere, and this constant is in $[0, 1]$.

Proof: We have for any $t > 0$ and $T \geq 0$,

$$\begin{aligned} S_{T+t}[F](x) - S_T[F](\phi^t x) &= \int_0^{T+t} f_s(\phi^s x) ds - \int_0^T f_s(\phi^{t+s} x) ds \\ &= \int_0^t f_s(\phi^s x) ds + \int_0^T (f_{s+t} - f_s)(\phi^{t+s} x) ds. \end{aligned}$$

Denote by $G^{(t)} = (g_s^{(t)})_{s\geq 0}$ the family of functions defined by

$$\forall s \geq 0, \quad g_s^{(t)} = f_s - f_{s+t}.$$

The above equation can be rewritten

$$S_{T+t}[F](x) - S_T[F](\phi^t x) = S_t[F](x) - S_T[G^{(t)}](\phi^t x).$$

Since F is decreasing, $G^{(t)}$ is positive, and

$$I_\infty[G^{(t)}] = I_t[F] < \infty.$$

From Lemma 1, $S_T[G^{(t)}](x)$ has a finite limit $H^{(t)}(x)$ as $T \to +\infty$ for almost every x, which is moreover in $L^1(\mu)$. So

$$\frac{S_t[F](x)}{I_T[F]} \geq \frac{I_{T+t}[F]}{I_T[F]}\frac{S_{T+t}[F](x)}{I_{T+t}(x)} - \frac{S_T[F](\phi^t x)}{I_T[F]} \geq -\frac{H^{(t)}(\phi^t x)}{I_T[F]}.$$

Since F is a decreasing family and $I_\infty[F] = +\infty$, it follows that $I_{T+t}[F]/I_T[F]$ goes to 1 as T goes to infinity. From the last equation, we deduce that for any x such that $H^{(t)}(x) < \infty$, we have

$$L(x) = L(\phi^t x).$$

If $t \leq 1$, then $H^{(t)} \leq H^{(1)}$. The sets $L^{-1}([a,b])$ are invariant under ϕ^t for any t in $[0,1]$ and $a < b$, apart from a set of 0 measure, because $H^{(1)}$ is finite almost everywhere. Because μ is ergodic, the sets $L^{-1}([a,b])$ are of measure 0 or 1. A dichotomy on the intervals $[a,b]$ then shows that L must be constant almost everywhere. The proof that l is constant almost everywhere is identical. Like in [KM99], by the Fatou Lemma

$$\int_X l(x)d\mu \leq \liminf_{T\to\infty}\left(\int_X S_T[F](x)d\mu(x)\right)/I_T[F] = 1. \qquad \blacksquare$$

Proof of Proposition 1: Let h be any function in $L^q(\mu)$. The family of integrals

$$J(T) = \int_X h(x)S_T[F](x)/I_T[F]d\mu(x)$$

is bounded, since $S_T[F]/I_T[F]$ is also bounded in L^p-norm. If one takes any sequence T_n that goes to infinity, such that $J(T)$ has a limit l, then by the Banach–Alaoglu Theorem one can find a subsequence $T_{\phi(n)}$ such that $S_T[F]/I_T[F]$ has a weak limit g in $L^p(\mu)$. From Lemma 2, $g = 1$. Then $l = \int_X hd\mu$ necessarily, and so $J(T)$ goes to $\int_X hd\mu$ as T tends to infinity. Since this is valid for any h in $L^q(\mu)$, this means exactly that $S_T[F]/I_T[F]$ converges weakly to 1 as T goes to infinity. On the other hand, Lemma 3 implies that $L(x) = \limsup_{T\to+\infty} S_T[F](x)/I_T[F]$ is constant almost everywhere. Let c be this constant. For a contradiction, suppose that $c < 1$. Then for any $\epsilon > 0$ sufficiently small, there exists a set $E \subset X$ of positive measure, such that for any x in E, and $T \geq 1/\epsilon$, we have $S_T[F](x) < (1-\epsilon)I_T[F]$. Fix such an ϵ, and let 1_E be the characteristic function of E. It belongs to $L^q(\mu)$ and so we have that $\int_E S_T[F]/I_T[F]d\mu$ goes to $\mu(E)$ as T goes to infinity. But we can also show that these integrals are less that $(1-\epsilon)\mu(E)$ provided that $T > 1/\epsilon$. This is a contradiction, and so $c \geq 1$. ∎

3. Proof of Theorem 1

The universal cover $\tilde{V}$ of V is homothetic to the real hyperbolic space $\mathbb{H}^d_{\mathbb{R}}$ of real dimension d, complex hyperbolic space $\mathbb{H}^d_{\mathbb{C}}$ of real dimension $2d$, quaternionic hyperbolic space $\mathbb{H}^d_{\mathbb{H}}$ of real dimension $4d$, or the Cayley hyperbolic plane $\mathbb{H}^2_{Ca}$ of real dimension 16 (see [Mo73] for a description of these spaces). We denote by δ the volume entropy of $\tilde{V}$, which is

$$\delta = \lim_{t\to\infty} \frac{\log(\mathrm{Vol}(B(o,t)))}{t}.$$

In fact, the volume of balls of radius t and the area of spheres of radius t are both equivalent to some constant times $\exp(\delta t)$ as $t \to +\infty$. It is not necessary for what follows to normalize the sectional curvature of $\tilde{V} = \mathbb{H}^d_{\mathbb{K}}$ between -4 and -1, but if this is the case and $d \geq 2$, $\delta = d-1 = n-1$ in the case $\mathbb{K} = \mathbb{R}$ and the curvature is -1, $\delta = n - 2 + \dim_{\mathbb{R}} \mathbb{K}$ if $\mathbb{K} \neq \mathbb{R}$.

We recall that $T^1\tilde{V} = \mathcal{G} \times \mathbb{R}$, where $\mathcal{G}$ is the space of oriented, unpointed geodesics, and the geodesic flow ϕ^t acts on $\phi^t(g,s) = (g, s+t)$; in case one writes $\mathcal{G} = \partial\tilde{V} \times \partial\tilde{V} - diag$, this is simply the *Hopf decomposition.* Let ν be the measure on $\mathcal{G}$ such that $\mu = \nu \otimes dt$. The action of the geodesic flow on T^1V is ergodic with respect to μ, and since V is of finite volume, we can freely normalize μ to be a probability measure.

The following two estimates will be the main ingredients of the proof. Note that the first one is the only point where the symmetric space assumption is used.

LEMMA 4: *There exists a constant $c_1 > 0$ such that, for any two balls in $\tilde{V}$ of respective center and radius (o,r), (o',r') that satisfy $r < 1$, $r' < 1$, and $d(o,o') > 2$, then the ν-measure of geodesics which intersects those two balls is less than*

$$c_1 r^{n-1} r'^{n-1} \exp(-\delta d(o,o')).$$

Proof: Let $R = d(o,o')$. Consider the hyperbolic sphere $S(o,R)$ centered on o of radius R. There exists a constant $c > 0$, independent of R, such that one can find $k \geq cr'^{n-1}\exp(\delta R)$ points $p_1, \ldots, p_k$ on S satisfying that the balls $B(p_i, 2r')$ of center p_i and radius $2r'$ are disjoints. Then the measures m_i of the geodesics which intersect both $B(o,r)$ and $B(p_i,r')$ are all equal, because there exists an isometry that sends o to o and p_i to p_j for all i,j (hyperbolic spaces are *2-points homogeneous*), and are equal to the measure m we are interested in. Let M be the measure of geodesics which intersect $B(o,r)$. Any geodesic intersecting $B(o,r)$ intersects at most two balls $B(p_i,r')$, so one has the inequality $km \leq 2M$. Since M is less than some constant times r^{n-1}, the result follows from the lower bound on k. ∎

LEMMA 5 ([Su79]): *For any discrete group Γ of isometries of $\tilde{V}$ such that $\Gamma\backslash\tilde{V}$ is of finite volume, for any p in V, there exists a constant $c_2 > 0$ such that for any $T \geq 0$,*

$$|\{\gamma \in \Gamma : d(p,\gamma p) \leq T\}| \leq c_2 \exp(\delta T). \quad ∎$$

The fundamental group $\Gamma = \pi_1(V)$ can be seen as a subgroup of the isometry group of $\tilde{V}$. For a point p in a Riemannian manifold W, let us write $i_W(p)$ for the injectivity radius of p in W. Given a real number $R > 0$, a ball B of center o and radius r, containing a point p, in a Riemannian manifold W, with $0 < r \leq R \leq i_W(p)/4$, we define for any v in T^1W,

$$H_B(v) = \begin{cases} 1 & \text{if } d(\pi(\phi^\theta v), o) \leq r, \text{ for some } \theta \in [-R/2, R/2], \\ 0 & \text{otherwise.} \end{cases}$$

From now on, we can (and we will) assume that the radius r_t of the ball B_t goes to 0 as t tends to infinity, because otherwise $\bigcap_{t\geq 0} B_t$ would contain a ball, and the result follows from Birkhoff's ergodic Theorem. Then, necessarily, $\bigcap_{t\geq 0} B_t$ is a point p in V. We fix R to be the minimum between $i_V(p)/4$ and 1. Moreover, one can assume that r_t is strictly less than R for any $t \geq 0$; it will be convenient to define r_t to be r_0 for all t negative. Let $\tilde{p}$ and $\tilde{B}_t$ be lifts of p, B_t, such that $\tilde{p}$ is contained in $\tilde{B}_t$ for all $t \geq 0$.

We define $f_t = H_{B_t}$ for this R and $W = V$, and $g_t = H_{\tilde{B}_t}$ for this R and $W = \tilde{V}$. Thus, the lift $\tilde{f}_t$ of f_t to $T^1\tilde{V}$ satisfies

$$\tilde{f}_t = \sum_{\gamma \in \Gamma} g_t \circ \gamma.$$

The family of functions $F = (f_t)_{t\geq 0}$ on T^1V is measurable, positive, and decreasing because the family of balls B_t is decreasing. Moreover, it can be seen easily that $\int_{T^1V} f_t d\mu$ is equivalent as $t \to +\infty$ to r_t^{n-1} up to a multiplicative constant depending on n and R only, and thus $I_T[F]$ is equivalent up to a multiplicative constant to $\int_0^T r_t^{n-1} dt$, provided any of these two functions tends to infinity as T goes to infinity.

The main argument is

LEMMA 6: *The L^2-norm of $S_T[F]/I_T[F]$ is bounded for all $T \geq 1$.*

Proof: Using Fubini's Theorem, one can write for any $T \geq 0$, any v in T^1V,

$$S_T[F](v)^2 = 2\int_0^T \int_0^s f_t(\phi^t v) f_s(\phi^s v) dtds.$$

Let us integrate over T^1V, and then make a change of variables $w = \phi^s v$. We obtain

$$\int_{T^1V} S_T[F](v)^2 dv = 2\int_0^T \int_0^s \int_{T^1V} f_s(w) f_t(\phi^{t-s} w) dwdtds.$$

Let D be a strict fundamental domain of $\tilde{V}$ for Γ, such that D contains the ball of center $\tilde{p}$, radius $3R$, and $T^1D = \tilde{\pi}^{-1}(D)$.

$$\int_{T^1V} S_T[F](v)^2 dv = 2\int_0^T \int_0^s \int_{T^1D} \tilde{f}_s(w) \tilde{f}_t(\phi^{t-s} w) dwdsdt.$$

The choice of D insures that for all w in T^1D and $s \geq 0$, we have $\tilde{f}_s(w) = g_s(w)$ because g_s has support contained in D, and so in the sum $\sum_{\gamma\in\Gamma} g_t(\gamma(w))$, all terms but the one corresponding to $\gamma = id$ are zero. So we have

$$\int_{T^1V} S_T[F](v)^2 dv = 2\int_0^T \sum_{\gamma\in\Gamma} \int_0^s \int_{T^1D} g_s(w) g_t(\gamma\phi^{t-s} w) dwdtds.$$

Putting $v = -w$, and using the symmetry of the functions g_t, we deduce that

$$\int_{T^1V} S_T[F](v)^2 dv = 2\int_0^T \int_{T^1D} \sum_{\gamma\in\Gamma} \int_0^s g_s(v) g_t(\gamma\phi^{s-t} v) dvdtds.$$

For any positive integer i, we define Γ_i to be the finite set of γ in Γ such that $d(\tilde{p}, \gamma\tilde{p})$ is in the interval $[i-1, i+1]$. Since the union of these sets is Γ itself, we have

$$\int_{T^1V} S_T[F](v)^2 dv \leq 2\int_0^T \sum_{i>0} \sum_{\gamma\in\Gamma_i} \int_0^s \int_{T^1D} g_s(v) g_t(\gamma\phi^{s-t} v) dvdtds.$$

Let us analyse when the integrated function can be strictly positive. For fixed γ, v, s and t, the quantity

$$g_s(v) g_t(\gamma\phi^{s-t} v)$$

is not zero if and only if there exists some $(\theta_1, \theta_2) \in [-R/2, R/2]^2$ such that $\pi(\phi^{\theta_1} v)$ is in $\tilde{B}_s$, and $\pi(\phi^{\theta_2+s-t} v)$ is in $\gamma^{-1}\tilde{B}_t$.

In this case, the quantity above is equal to 1 and one can state, in virtue of the triangle inequality, that

$$(1) \qquad |(s-t) - d(\tilde{p}, \gamma\tilde{p})| < |\theta_1| + |\theta_2| + 2r_t + 2r_s \leq 5R,$$

and moreover that the geodesic ray $(\phi^u v)_{u \geq -1}$ intersects the ball of center $\gamma^{-1}\tilde{p}$ and radius $2r_{s-d(\tilde{p},\gamma\tilde{p})-5R}$ (this is because r_t is decreasing), and intersects also the ball of center $\tilde{p}$ and radius $2r_s$.

So, for fixed s, v and γ in Γ_i, Equation (1) implies that $g_s(v)g_t(\gamma\phi^{s-t}v)$ is zero when t is outside the interval $[s - d(\tilde{p}, \gamma\tilde{p}) - 5R, s - d(\tilde{p}, \gamma\tilde{p}) + 5R]$, of length $10R$. So we have

$$(2) \qquad \int_0^s g_s(v)g_t(\gamma\phi^{s-t}v)dt \leq 10R,$$

in all cases, and this integral vanishes if the geodesic generated by v does not intersect the two balls $B(\tilde{p}, 2r_s)$, $B(\gamma^{-1}\tilde{p}, 2r_{s-i-5R-1})$, or if $s < i - (5R+1)$. If moreover $i \geq 5R+3$, then $s - t \geq 2$ by (1), and by Lemma 4 the measure of the set of such geodesics is bounded by $c_1 4^{n-1} r_s^{n-1} r_{s-i-5R-1}^{n-1} e^{-\delta(i-1)}$. Since on a geodesic, the interval of vectors v such that $g_s(v) > 0$ is at most of length 3, we obtain when s and $\gamma \in \Gamma_i$ are fixed and $i \geq 5R+3$,

$$\int_{T^1D} \int_0^s g_s(v)g_t(\gamma\phi^{s-t}v)dtdv \leq 30R4^{n-1}c_1 r_s^{n-1} r_{s-i-5R-1}^{n-1} e^{-\delta(i-1)},$$

and this quantity is zero if $s + 5R + 1 < i$. Thus, if $i \geq 5R+3$, summing over γ in Γ_i, we obtain

$$\sum_{\gamma\in\Gamma_i} \int_{T^1D} \int_0^s g_s(v)g_t(\gamma\phi^{s-t}v)dtdv \leq 30R4^{n-1}c_1 r_s^{n-1} r_{s-i-5R-1}^{n-1} e^{-\delta(i-1)} N_i,$$

where we have written N_i for the number of elements of Γ_i; moreover, the majorated sum does in fact vanish if $s + 5R + 1 < i$. From Lemma 5, $N_i \leq c_2 e^{\delta(i+1)}$. Let us write $[x]$ for the integer part of any real number x. It follows that there exists some positive constant c_3 (depending only on R) such that

$$\int_0^T \sum_{i>0} \sum_{\gamma\in\Gamma_i} \int_0^s \int_{T^1D} g_s(v)g_t(\gamma\phi^{s-t}v)dvdtds$$
$$\leq c_3 \int_0^T \left(\sum_{i=[5R+4]}^{[s+5R]+1} r_s^{n-1} r_{s-i-5R-1}^{n-1} \right) ds$$

$$+\int_0^T \sum_{i=1}^{[5R+3]} \sum_{\gamma\in\Gamma_i} \int_{T^1D} \int_0^s g_s(v) g_t(\phi^{s-t}v) dt dv ds.$$

Observing that $\bigcup_{i=1}^{[5R+3]} \Gamma_i$ is a finite set, we write K its cardinal. Since (r_t) is decreasing, $r_{s-i-5R-1}^{n-1} \le \int_{s-i-5R-2}^{s-i-5R-1} r_t^{n-1} dt$. Moreover, in Equation (2), we can freely multiply the upper bound by $g_s(v)$, because if $g_s(v) = 0$ then the majorated integral vanishes, else $g_s(v) = 1$. This allows us to write

$$\begin{aligned}\int_{T^1V} S_T[F](v)^2 dv \le & 2c_3 \int_0^T r_s^{n-1} \sum_{i=[5R+4]}^{[s+5R+1]} \int_{s-i-5R-2}^{s-i-5R-1} r_t^{n-1} dt ds \\ & + 2K \int_0^T \int_{T^1D} 10R g_s(v) dv.\end{aligned}$$

Since $\int_{T^1D} g_s(v) dv$ is equivalent up to a multiplicative constant to r_s^{n-1} as s tends to infinity, there exists some positive constant c_4, depending only on R, such that

$$\begin{aligned}\int_{T^1V} S_T[F](v)^2 dv &\le 2c_3 \int_0^T r_s^{n-1} \int_{-10R-4}^{s-10R-4} r_t^{n-1} dt ds + c_4 \int_0^T r_t^{n-1} dt \\ &\le 2c_3 \int_0^T r_s^{n-1} \int_{-10R-4}^{s} r_t^{n-1} dt ds + c_4 \int_0^T r_t^{n-1} dt,\end{aligned}$$

and by Fubini's Theorem,

$$\int_{T^1V} S_T[F](v)^2 dv \le c_3 \Big(\int_0^T r_s^{n-1} ds \Big)^2 + (2(10R+4)c_3 r_0 + c_4) \Big(\int_0^T r_s^{n-1} ds \Big).$$

Since $I_T[F]$ is equivalent up to a multiplicative constant to $\int_0^T r_t^{n-1} dt$ as T tends to infinity, we conclude that $S_T[F]/I_T[F]$ is bounded in L^2-norm. ∎

Proof of Theorem 1:

CONVERGENCE CASE: We assume $\int_0^{+\infty} r_t^{n-1} dt < +\infty$. For any $w \in T^1V$ and $s \ge 0$, if $\pi(w)$ is in B_s, then from the definition of f_s we have

$$\int_{-R}^{R} f_s(\phi^t w) dt \ge R.$$

Since $F = (f_s)_{s\ge 0}$ is decreasing, we can write, after a change of variable $u = t + s - R$, that

$$\int_{s-2R}^{s} f_u(\phi^{u-s+R} w) du \ge R,$$

and so if $\pi(\phi^s v)$ is in B_s, we have

$$\int_{s-2R}^{s} f_t(\phi^{t+R}v)dt \geq R.$$

However, Lemma 1 states that for almost every v,

$$\int_0^\infty f_t(\phi^t v)dt < \infty,$$

and so for a set of full measure, we can rewrite it

$$\int_0^\infty f_t(\phi^t(\phi^R v))dt < \infty,$$

and by the preceding assertion, this implies that for t sufficiently large and v in a set of full measure, $\pi(\phi^t v)$ is not in B_t.

DIVERGENCE CASE: We assume $\int_0^{+\infty} r_t^{n-1}dt = +\infty$. Let v in T^1V be such that there exists some $T > 0$, such that for all $t \geq T$, $\pi(\phi^t v)$ is not in B_t. Since the balls B_t are decreasing, we have that for any $s \geq t \geq T$, $\pi(\phi^t v) \notin B_s$. In particular, if $t \geq T + R$ and $t \in [s, s + R]$, $\pi(\phi^{t-R}v) \notin B_s$. From the definition of f_s, this implies that $f_s(\phi^{s-R/2}v) = 0$, for all $s \geq T + R$. This shows that $S_\infty[F](\phi^{-R/2}v) < +\infty$, provided the geodesic ray $(\phi^t v)_{t\geq 0}$ intersects B_t for bounded times t only. But from Lemma 6 and Proposition 1, we have that $S_\infty[F](v) = +\infty$ for almost every v. So the preceding situation may happen for a set of v of null measure only. ∎

Proof of Corollary 1: We consider the family of balls centered on p of radius $r_t = (t + 1)^{-1/(n-1)-\epsilon}$ with arbitrary small $\epsilon > 0$, and use the convergence case of Theorem 1 to obtain the upper bound $1/(n - 1) + \epsilon$. Then we use the divergence case applied to $r_t = (t + 1)^{-1/(n-1)}$ in order to obtain the lower bound $1/(n - 1)$. ∎

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